The Weighted Cardinality Estimation Problem

Aviv Yehezkel

Ph.D. Seminar

Supervisors: Prof. Reuven Cohen and Dr. Liran Katzir

Overview

- Cardinality Estimation Problem

- Weighted Cardinality Estimation Problem

- The Unified Scheme

Cardinality Estimation Problem

Motivation

Given a very long stream of elements with repetitions,

How many are distinct?

Cardinality of a Stream

- Let M be a stream of elements with repetitions
  - N is the number of elements called the size
  - n the number of distinct elements called cardinality

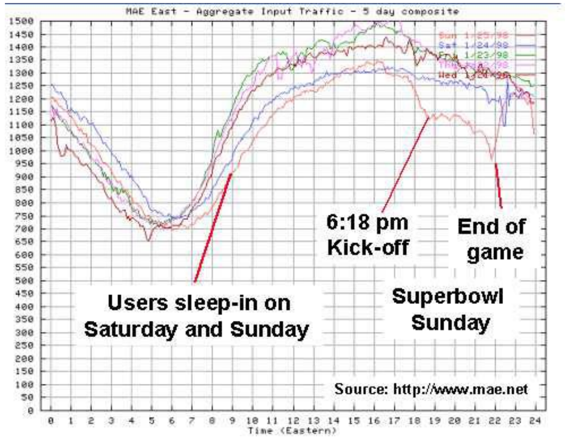
$$\boldsymbol{N = 8}$$
$$\boldsymbol{n = 4}$$

Element	Multi
C	1
D	3
B	3
Z	1

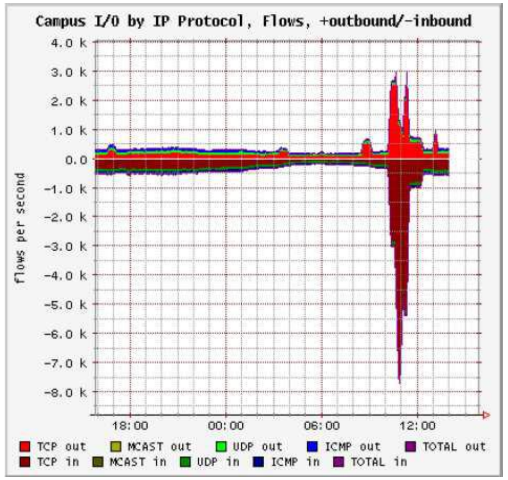
- **The problem:**

  compute the cardinality n in one pass and with small fixed memory

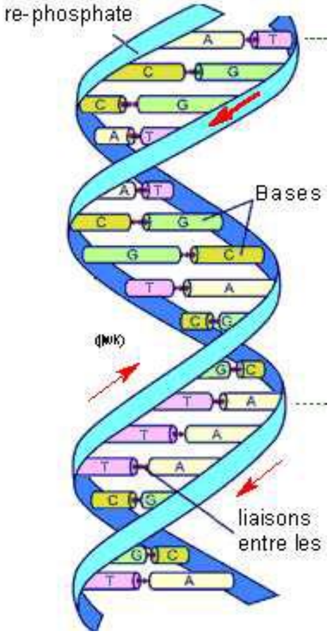
Many Applications



Traffic analysis



Attacks detection



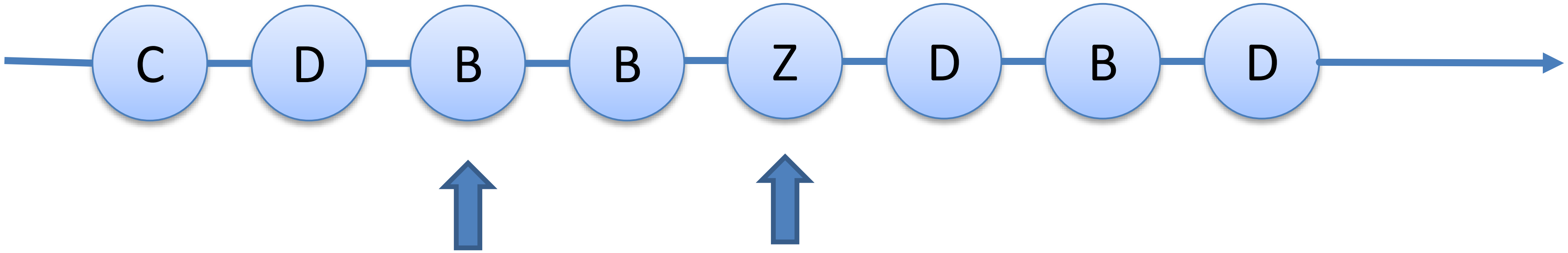
Genetics

Linguistic

and more…

Exact Solution

- Maintain distinct elements already seen



counter = 3 counter = 4

- One pass, but memory in order of n
- Lower bound: $\Omega(n)$ memory needed

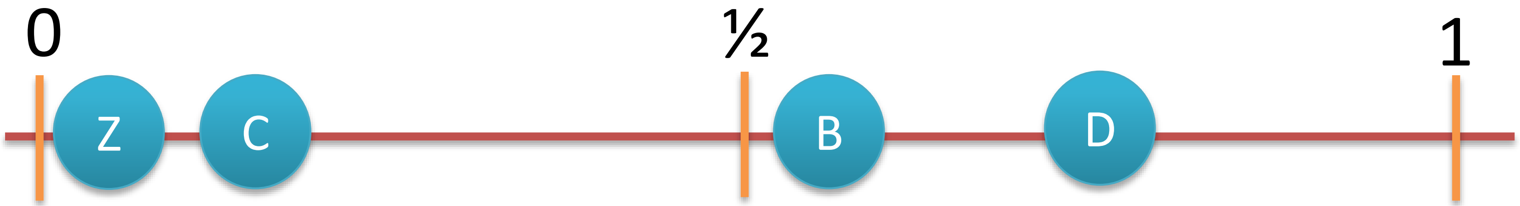
Probabilistic Solution

- Main idea:
  - relax the constraint of exact value of the cardinality
  - An estimate with good precision is sufficient for the applications

- Several algorithms:
  - Probabilistic counting
  - HyperLogLog
  - Linear Counting
  - Min Count
  - ….

Probabilistic Solution

- Elements of M are hashed to random variables in (0,1)

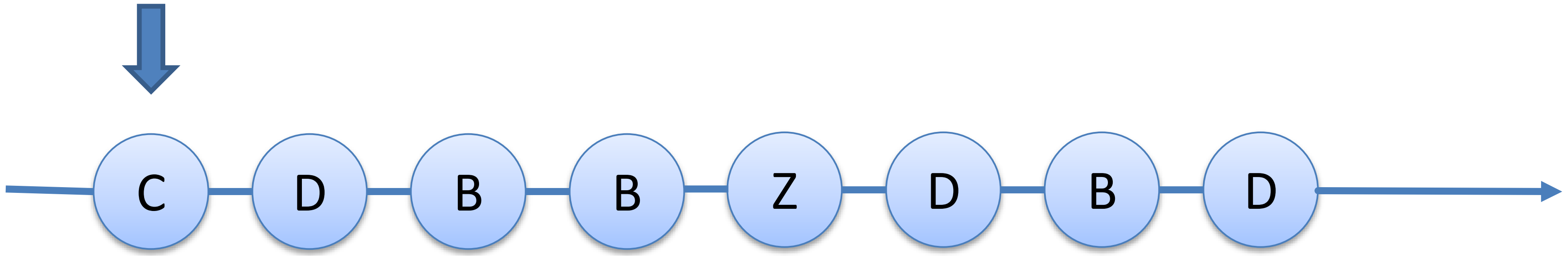


- Idea: use the maximum\minumum to estimate the cardinality
  - One pass
  - Constant memory

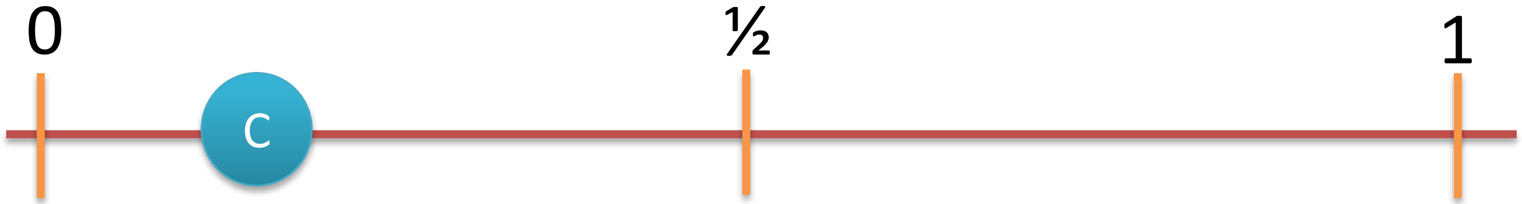
Probabilistic Solution

- Elements of M are hashed to random variables in (0,1)
- Intuition:
    - If there are 10 distinct elements,
    - Expect the hash values to be spaced about $\frac{1}{10}$th apart from each other
- $\mathbb{E}(max) = \frac{n}{n+1}$

Probabilistic Solution

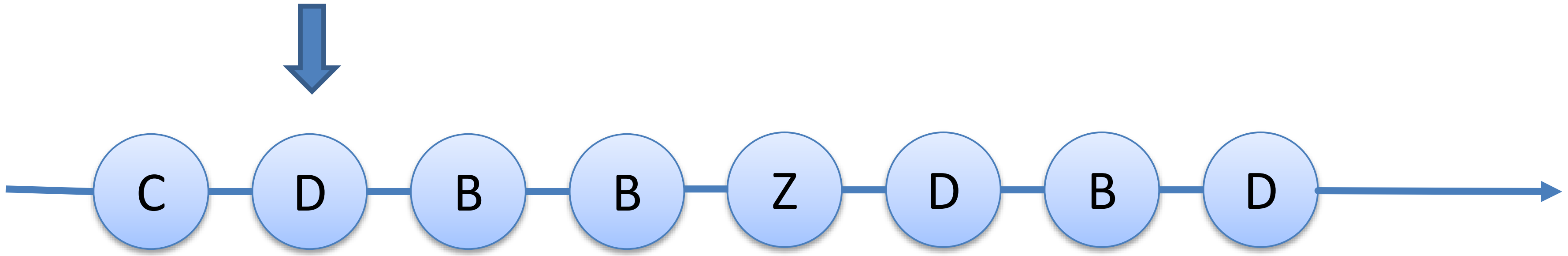


$h(C) = 0.347$

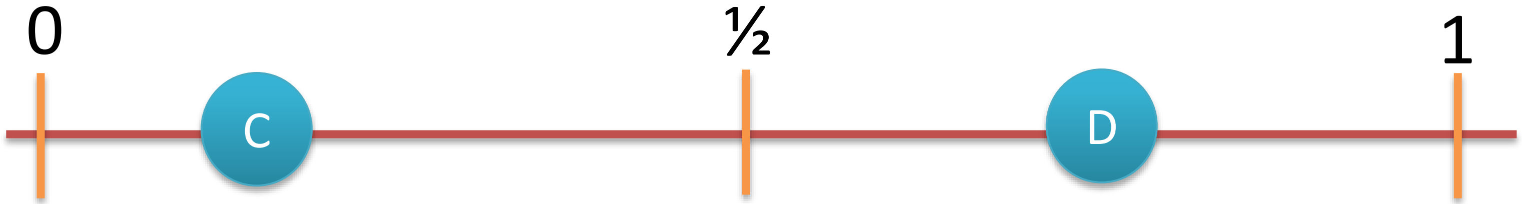


$$h^{+} = 0.347$$

Probabilistic Solution

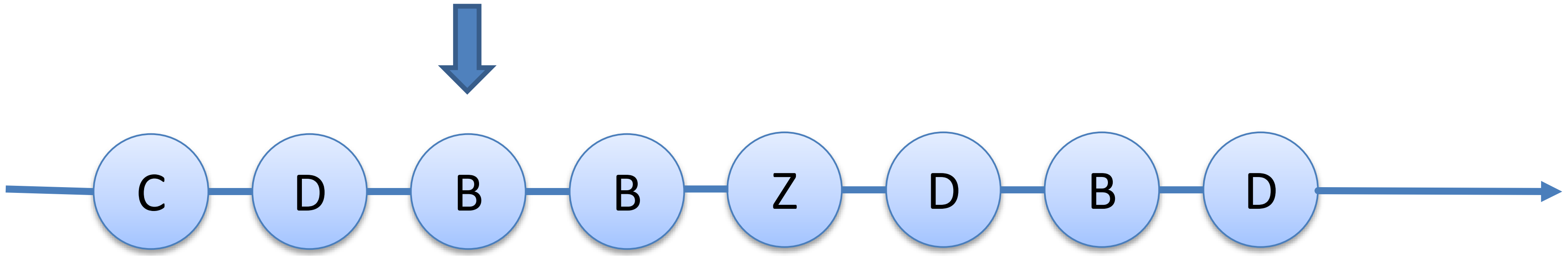


$$h(D) = 0.773$$

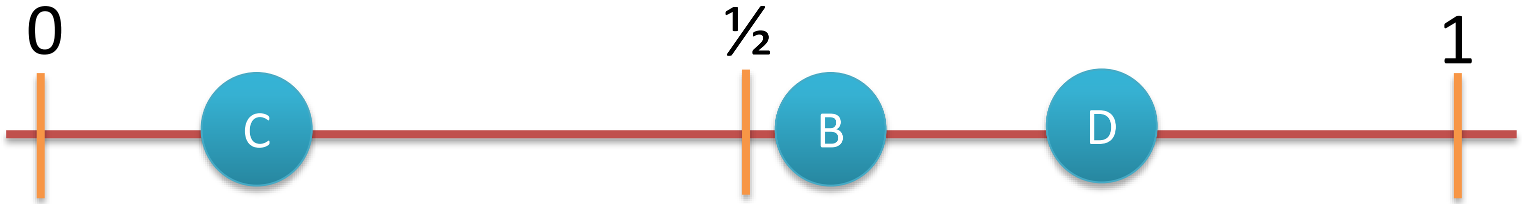


$$\boxed{h^{+} = 0.773}$$

Probabilistic Solution

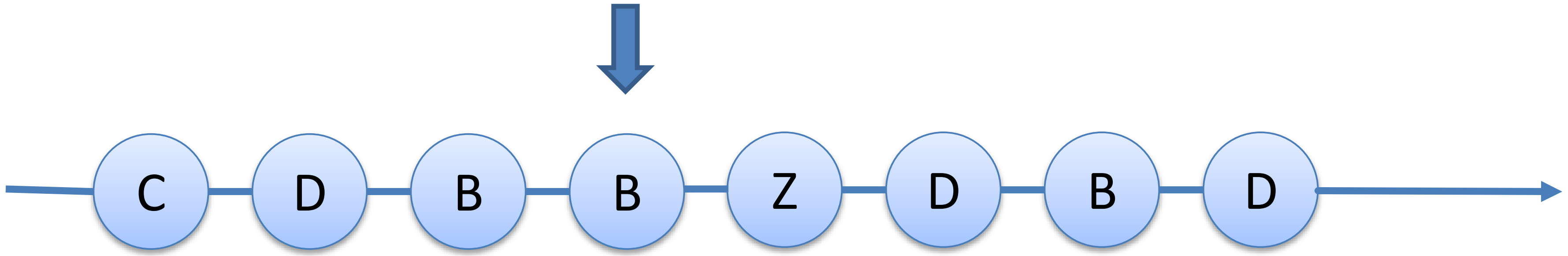


$h(B) = 0.512$

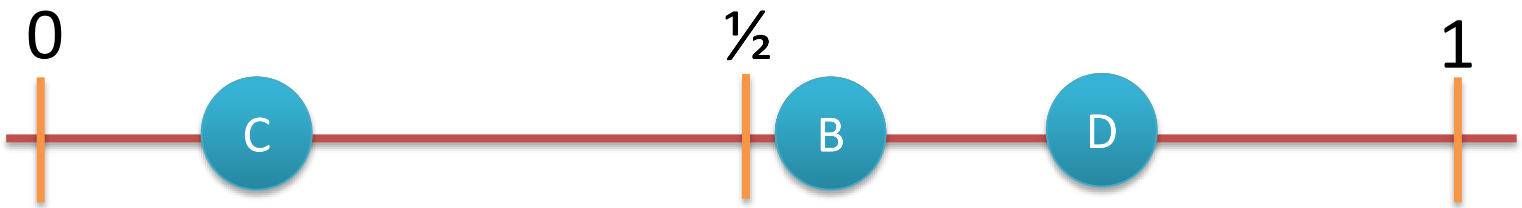


$$\boxed{h^{+} = 0.773}$$

Probabilistic Solution

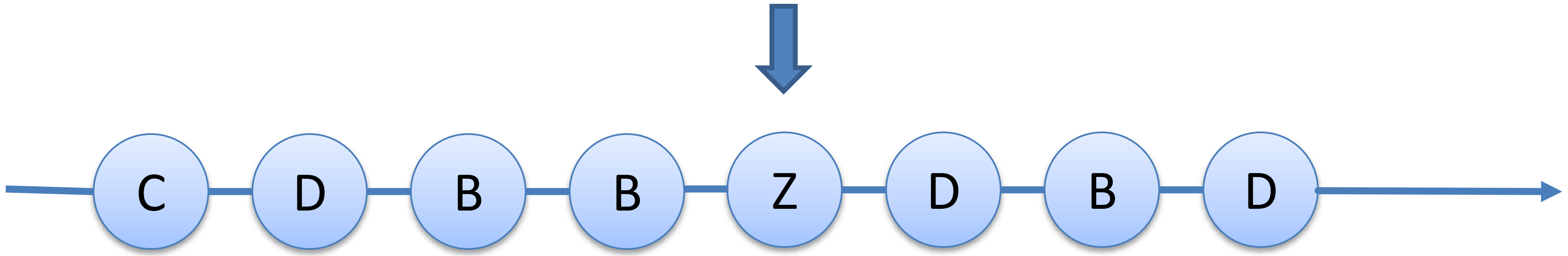


$h(B) = 0.512$

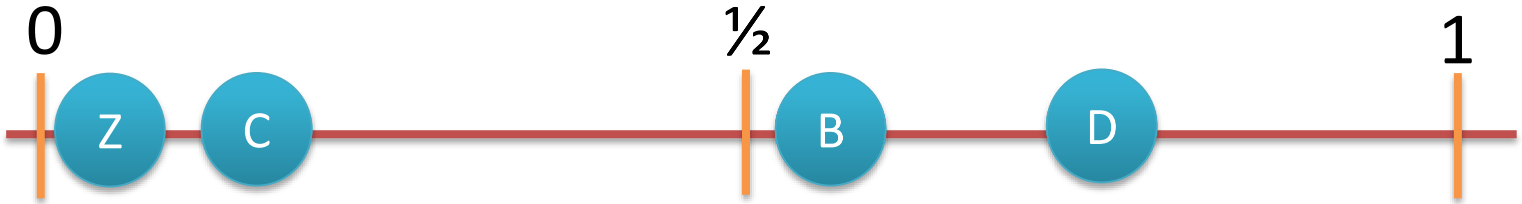


$$\boxed{h^{+} = 0.773}$$

Probabilistic Solution

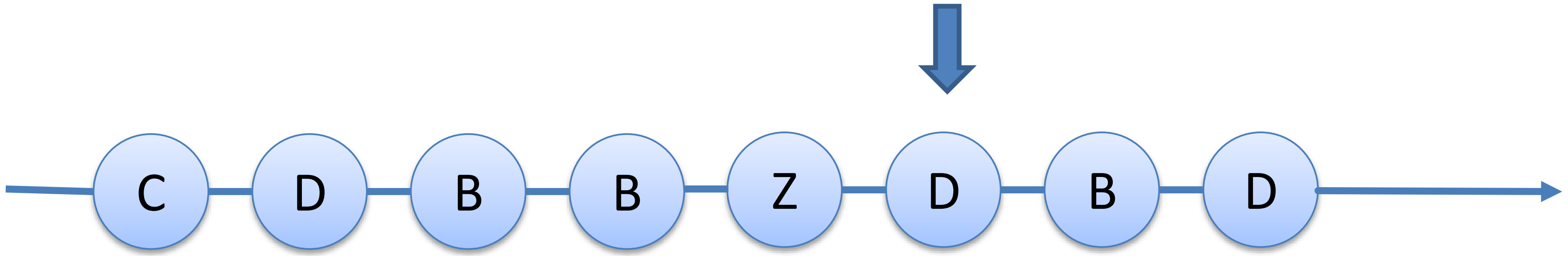


$h(Z) = 0.139$

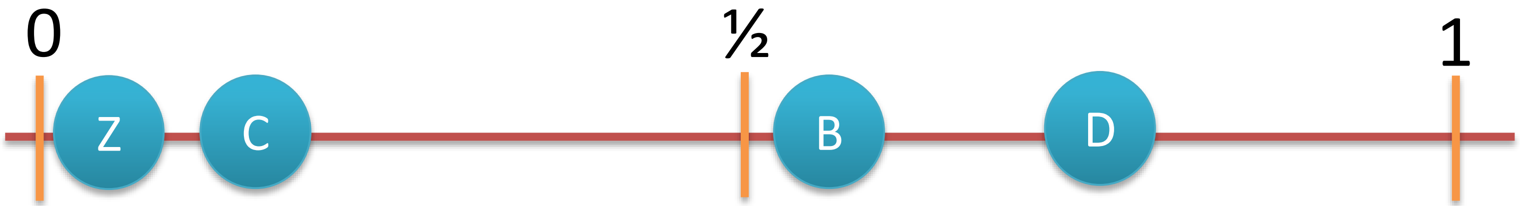


$$\boxed{h^{+} = 0.773}$$

Probabilistic Solution

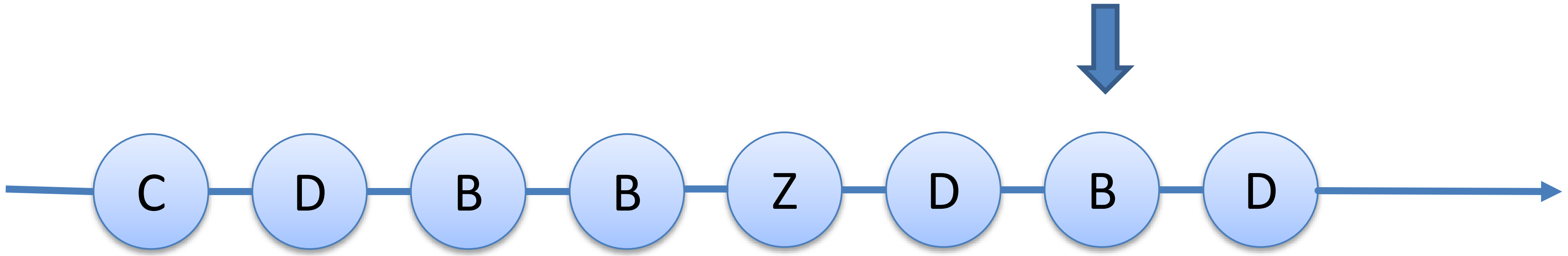


$h(D) = 0.773$

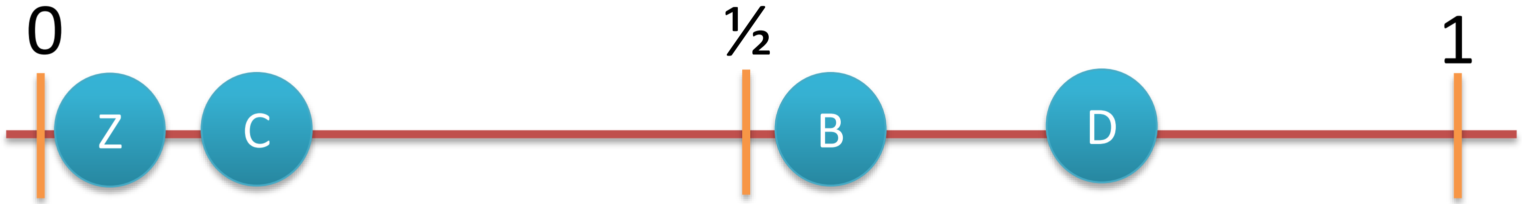


$$\boxed{h^{+} = 0.773}$$

Probabilistic Solution

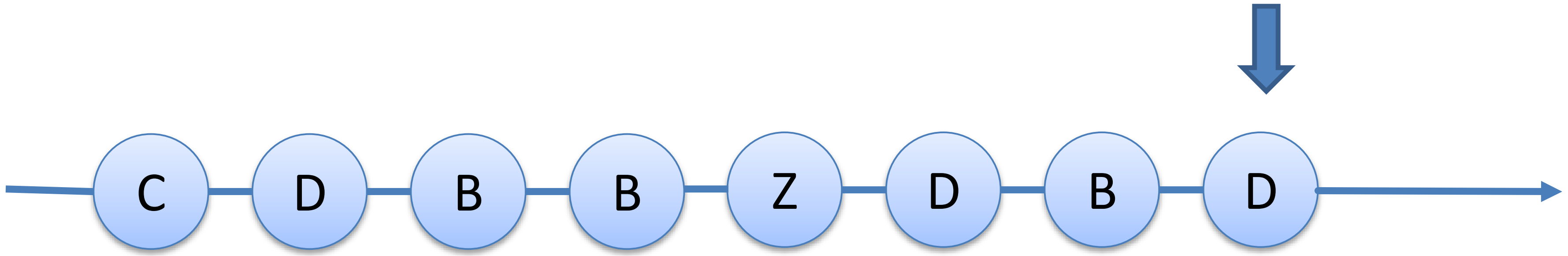


$h(B) = 0.512$

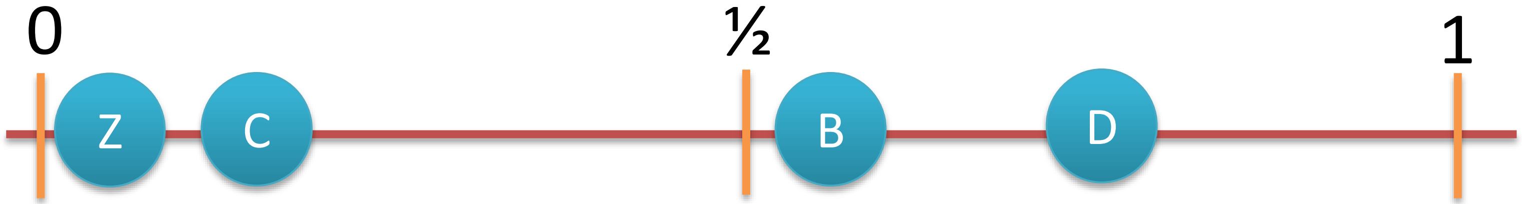


$$\boxed{h^{+} = 0.773}$$

Probabilistic Solution

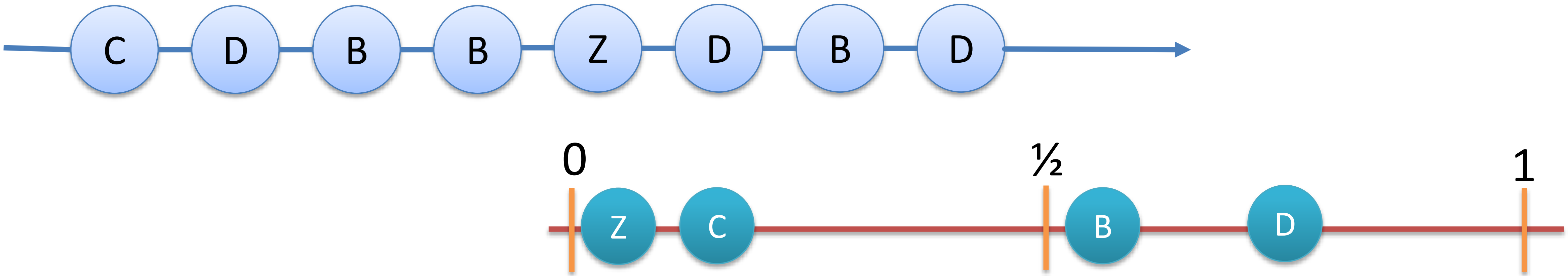


$h(D) = 0.773$



$$\boxed{h^{+} = 0.773}$$

Probabilistic Solution



- $\mathbb{E}(max) = \frac{n}{n+1} = 0.773$
- Estimated cardinality = 3.405
- Actual cardinality = 4

Chassaing Algorithm

- Simulate m different hash functions
  - m maxima $h_1^+, h_2^+, \ldots, h_m^+$

- Estimate $= \dfrac{m-1}{\sum(1-h_k^+)}$

Chassaing Algorithm

- $h_k^+ \sim \frac{n}{n+1}$

- $\sum(1 - h_k^+) \sim \sum \frac{1}{n+1} = \frac{m}{n+1}$

- Therefore,
  - Estimate $= \frac{m-1}{\sum(1 - h_k^+)} \sim n$

Chassaing Algorithm

- Relative error $\approx 1\,/\sqrt{m}$ for a memory of m words
- Minimal variance unbiased estimator (MVUE)

Formal Definition

Instance:

A stream of elements $x_1, x_2, \ldots, x_s$ with repetitions, and an integer m

Let n be the number of different elements, denoted by $e_1, e_2, \ldots, e_n$

Objective:

Find an estimate $\hat{n}$ of n, using only m storage units, where $m \ll n$

Min/Max Sketches

- Use m different hash functions
- Hash every element x_i to m uniformly distributed hashed values $h_k(x_i)$
- Remember only the minimum/maximum value for each hash function h_k
- Use these m values to estimate n

Generic Max Sketch Algorithm

Algorithm 1

1. Use m different hash functions
2. For every h_k and every input element x_i, compute $h_k(x_i)$
3. Let $h_k^+ = \max\{h_k(x_i)\}$ be the maximum observed value for h_k
4. Invoke $ProcEstimate(h_1^+, h_2^+, \ldots, h_m^+)$ to estimate n

Weighted Cardinality Estimation Problem

Weighted Sum of a Stream

- Each element is associated with a weight
- The goal is to estimate the weighted sum w of the distinct elements

Element	Weight
C	0.5
D	0.25
B	1
Z	1.25

$$\boldsymbol{w} = \sum \boldsymbol{w\_i} = $$
$$= \mathbf{0.5} + \mathbf{0.25} + \mathbf{1} + \mathbf{1.25} = \mathbf{3}$$

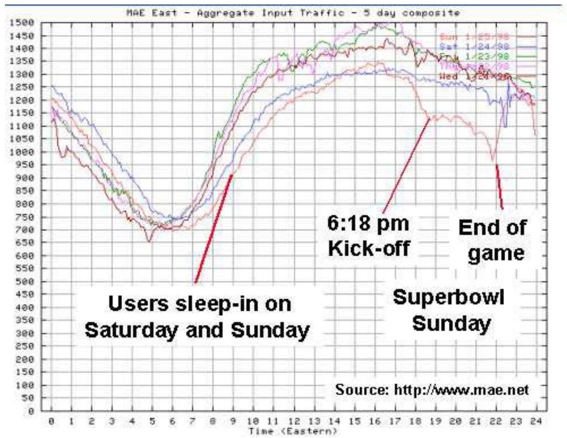
Application Example

- Stream of IP packets received by a server $x_1, x_2, \dots, x_s$

- Each packet belongs to a flow (connection) $\mathrm{e}_1, e_2, \dots, e_n$

- Each flow e_j imposes a load w_j on the server

- The weighted sum $w = \sum w_j$ represents the total load imposed on the server



Formal Definition

Instance:

A stream of weighted elements $x_1, x_2, \ldots, x_s$ with repetitions, and an integer m

Let n be the number of different elements, and let w_j be the weight of e_j

Objective:

Find an estimate $\hat{w}$ of $w = \sum w_j$, using only m storage units, where $m \ll n$

Our Contribution

- A unified scheme for generalizing any min/max estimator for the unweighted cardinality estimation problem to an estimator for the weighted cardinality estimation problem.

The Unified Scheme

Observation

- All min/max sketches can be viewed as a two step computation:
  1. Hash each element uniformly into (0, 1)
  2. Store only the minimum/maximum observed value

The Unified Scheme

- In the unified scheme we only change step (1) and hash each element into a Beta distribution.

- The parameters of the Beta distribution are derived from the weight of the element.

Beta Distribution

Lemma:

Let $z_1, z_2, \dots, z_n$ be independent RVs, where $z_i \sim Beta(w_i, 1)$

Then,

$$\max\{z_i\} \sim Beta(\textstyle\sum w_i, 1)$$

Corollary

- For every hash function,

$$h_k^+ = \max\{h_k(x_i)\} \sim \max\{U(0,1)\}$$
$$\sim \max\{Beta(1,1)\} \sim \mathrm{Beta}(\mathrm{n}, 1)$$

- Thus, estimating the value of n by Algorithm 1, is equivalent to estimating the value of α in the $\mathrm{Beta}(\alpha, 1)$ distribution of h_k^+

The Unified Scheme

For estimating the weighted sum:

- Instead of associating each element with a uniform hashed value
    - $h_k(x_i) \sim U(0,1)$
- We associate it with a RV taken from a Beta distribution
    - $h_k(x_i) \sim Beta\left(w_j, 1\right)$
    - w_j is the element's weight

Generic Max Sketch Algorithm - Weighted

<u>Algorithm 2</u>

- Use m different hash functions
- For every h_k and every input element x_i:
  1. compute $h_k(x_i)$
  2. transform to $h_k^{\wedge}(x_i) \sim Beta(w_j, 1)$
- Let $h_k^{+} = \max\{h_k^{\wedge}(x_i)\}$ be the maximum observed value for h_k
- Invoke $ProcEstimate(h_1^{+}, h_2^{+}, \dots, h_m^{+})$ to estimate the value of w

The Unified Scheme

- Practically, if

$$h_k(x_i) \sim U(0,1)$$

- Then,

$$h(x_i)_k^{1/w_j} \sim Beta(w_j, 1)$$

Distributions Summary

Unweighted	$h_k^+ \sim Beta(n, 1)$
Weighted	$h_k^+ \sim Beta(\mathrm{w} = \sum w_j, 1)$

The Unified Scheme

- The same algorithm that estimates n in the unweighted case can estimate w in the weighted case

- $ProcEstimate()$ is exactly the same procedure used to estimate the unweighted cardinality in Algorithm 1

The Unified Scheme Lemma

Estimating w by Algorithm 2 is equivalent to estimating n by Algorithm 1.

Thus, Algorithm 2 estimates w with the same variance and bias as that of the underlying procedure used by Algorithm 1.

Weighted Generalization for Chassaing Algorithm

- Estimate = $\dfrac{m-1}{\sum(1-h_k^+)}$

- But now,

$$h_k^+ = \max\{h_k^\wedge(x_i)\} = \max\{h_k(x_i)^{1/w_j}\}$$

Stochastic Averaging

- Presented by Flajolet in 1985
- Use 2 hash functions instead of m
- Overcome the computational cost at the price of negligible statistical efficiency in the estimator's variance

Stochastic Averaging

- Use 2 hash functions:
  1. $H_1(x_i) \sim \{1,2,\ldots,m\}$
  2. $H_2(x_i) \sim U(0,1)$

- Remember the maximum observed value of each bucket

- The generalization to weighted estimator is similar

Simulation

- We simulate a stream of weighted elements:
  - n elements from r weight classes
  - Each class is associated with a different weight, $w_j \in [w_{min}, w_{max}]$

- Weights distributions:
  - Uniform distribution: $f_j = \frac{1}{r}$
  - Normal distribution around $\frac{1}{2}(w_{min} + w_{max})$

Method-1 (Benchmark)

- We simulate a new stream of w unweighted elements $e_1, e_2, \ldots, e_w$
- The cardinality of the new stream is equal to the weighted sum w
- We then run the unweighted algorithm, without weighted adaptation

Method-2 (Unified Scheme)

- We apply our unified scheme and generalize the unweighted algorithm into a weighted algorithm

- We then run it on the original weighted input stream

Results – Chassaing Algorithm

weight parameters		bias		variance ratio
distribution	#classes	Method-1	Method-2	
uniform	16	0.00011	0.00046	0.986
uniform	64	0.00131	0.00056	1.024
uniform	512	0.00247	0.00134	0.979
normal	16	0.00252	0.00119	1.014
normal	64	0.00048	0.00432	1.006
normal	512	0.00274	0.00051	0.995

$n = 100{,}000$; $m = 32$; 10,000 runs

Conclusion

- We showed how to generalize every min/max sketch to a weighted version
- The proposed unified scheme uses the unweighted estimator as a black box, and manipulates the input using properties of the Beta distribution
- We proved that estimating the weighted sum by our unified scheme is statistically equivalent to estimating the unweighted cardinality

Questions?

Thank You